1. Details of Stellar Evolution of Low Mass Stars

This is an outline of some issues discussed, together with the figures presented in class.

2. Gravitational Settling and Radiative Levitation

Gravitational settling (diffusion) - heavy elements sink towards center of star if gas in star is not well mixed. This is a diffusion process. Timescale for this to happen is long, several Gyr. Convection, when it exists within a star, is very effective in mixing gas from different layers within the convection zone. He abundance in outer regions of the Sun is reduced compared to center.

Radiative levitation - radiation pressure in absorption lines arising from a specific ion lifts that ion within the atmosphere. Calculations depend on the details of the electronic levels of the ion, the strength of its absorption coefficients, etc., so that this effect varies in importance a lot from ion to ion. More important in more luminous stars.
Fig. 1.— Gravitational settling (see panel for He) and radiative levitation (see panel for Fe and Ca) of selected elements in horizontal branch stars in M13. The horizontal axis is $T_{\text{eff}}$; the effects are not detectable for the cooler HB stars, and seen only in the hotter ones. The dashed horizontal lines represent the initial abundance for each element in the old globular cluster M13. (Source: Behr, Cohen, McCarthy..., 1999, ApJL)
Fig. 2.— The distribution of rotational velocity along the horizontal branch in the old globular cluster M13. The hottest stars have much lower rotational velocities than do the cooler ones. Rotation sets up internal currents which can keep the stellar interior and atmosphere well mixed so that diffusion or radiative levitation cannot happen. (Source: Behr, Cohen, McCarthy..., 1999, ApJL)
Fig. 3.— Predictions for an old (age $\sim 12$ Gyr) $0.8 \, M_\odot$ star of the effects of gravitational settling and radiative levitation in main sequence turnoff stars. (Source: Korn et al, A&A, Diffusion in NGC 6397, 2007)
3. Star formation and Pre-Main Sequence Evolution

3.1. Phases of evolution for pre-main sequence stars

dense, cool cloud in ISM collapses (fragments) to produce a protostar (Jeans mass
criterion for collapse)

Protostar collapses (free-fall) very quickly until H and He become ionized inside star

Opacity rises, now have proto-star with temperature gradient, hydrostatic equilibrium,
fully convective but too cool even in center for nuclear reactions. Collapse continues, star
fully convective. Model as fully convective star ($n = 1.5$ polytrope) with thin radiative
envelope to derive Hayashi track.

Core becomes radiative before $T_c$ reaches nuclear ignition level ($\sim 1.5 \times 10^6$ K). further
collapse proceeds along the Henyey track

(if $M \lesssim 0.5 M_\odot$, stellar core does not become radiative before $T_c$ reaches nuclear
ignition, the star reaches the main sequence without ever switching from the Hayashi track
onto the Henyey track.

Pre-MS lifetimes are short, and it is unusual to find such stars unless a) environment
is very young cluster and b) the mass of the protostar is low, which makes the pre-MS
timescales longer. Such stars show up as variable, mass ejecting stars, often surrounded by
lots of dust and gas and perhaps a proto-planetary disk which absorbs light and reradiates
it in the IR. They are often X-ray emitters as well.

if $M \lesssim 0.05 M_\odot$ the stellar core becomes degenerate before $T_c$ reaches the level for
nuclear ignition. This defines the lower limit in mass for a star.

The physical effect that sets the upper limit for the mass of a star is not as clear. We
use the criterion $P(rad)/P = 50\%$ in the center of the star to deduce a value of $\sim 100 \ M_\odot$. 

This is close to the mass of the most massive stars known in our galaxy.

### 3.2. Selected Results Derived in Class

We ignore accretion after the initial formation of the protostar throughout.

Jean’s mass: \( M_J \propto T^{3/2}/n^{1/2} \) where \( T \) and \( \rho \) are the values for the initial homogenous massive cool cloud.

Free-fall time for first collapse phase,

\[
t_{ff} = \left[ \frac{32 G \rho_0}{3 \pi} \right]^{-1/2}
\]

This is a short time, 4700 years for a number density in the original cloud of 1 atom/cm\(^3\), independent of the mass of the gas in the collapsing cloud fragment.

When ionization is complete, using the virial theorem we derive:

\[
\frac{R}{R_{\odot}} = \frac{43.2}{(1 - 0.2X)} \frac{M}{M_{\odot}} \quad \text{and} \quad T_c \approx 300,000 \mu (1 - 0.2X) K
\]

Gravitational collapse with hydrostatic eq. and no nuclear burning, timescale is set by luminosity

\[
\Delta t \approx \Delta E/L = -(1/2)\Delta \Omega/L \approx \frac{M^2 G}{2RL} \approx 1.6 \times 10^7 \frac{M}{M_{\odot}} \frac{R_{\odot}}{R} \frac{L_{\odot}}{L} \text{ years}
\]

Hayashi track: We use the polytrope solution for a fully convective star (perfect gas, adiabatic, \( n = 1.5 \)), matched to a thin radiative envelope to determine that
\[ \log(T_{\text{eff}}) = \alpha \log(L) + \beta \log(M) + \delta \]
where \( \alpha \), \( \beta \) and \( \delta \) are determined from the polytrope index \((n = 1.5)\) and the envelope opacity only.

\( \alpha \) is small, \(< 0.1 \) and \( \beta \sim 0.2 \). Thus the Hayashi line has \( d[\log(L)]/d[\log(T_{\text{eff}})] > 10 \) (very steep vertical line in HR diagram, as a pre-main sequence star of initial (and fixed) mass \( M \) contracts and its radius decreases, the star moves to lower \( L \) along the Hayashi track. \( d[\log(T_{\text{eff}})]/d[\log(M)] \sim 0.2 \), so higher mass pre-MS stars have their almost vertical tracks shifted towards higher \( T_{\text{eff}} \).

This result is independent of the source of stellar energy. The Hayashi line location in the CMD is determined by surface properties/surface boundary conditions. The extended outer layers fix the adiabat that describes the inner \( T(r) \) profile. At any stage, \( T_{\text{eff}} < T(\text{Hayashi line}) \) for a given \( L, M \) is forbidden.

Henyey track: now radiative core, star hotter, opacity lower, model star as totally radiative without a strong density gradient, use linear density model to get

\[
\frac{L}{L_\odot} \propto \left( \frac{M}{M_\odot} \right)^{5.5} \left( \frac{R_\odot}{R} \right)^{0.5}
\]

This defines the Henyey track as sloping towards higher \( T_{\text{eff}} \) and towards higher \( L \) from the end of the Hayashi track. The Henyey track evolution terminates when H burning begins in the pre-main sequence star, i.e. the star reaches the main sequence.
Fig. 4.— Upper panel: Central $T$ in units of $10^6$ K as a function of central density for a collapsing cloud of hydrogen for a collapsing mass of $(1/16)M_\odot$. Lower panel: $P(\text{rad})/P(\text{gas})$ in the center of a star as a function of mass. The maximum $T$ is barely adequate to support nuclear reactions. Degeneracy is becoming important at the highest $\rho$, causing the decrease of $T_c$ with increasing $\rho$. (Source: book by Phillips)
Fig. 5.— Pre-main sequence tracks for stars with masses from 0.5 to 15 $M_{\odot}$. The time required to reach the numbered points on each track, which correspond to specific stages in the pre-main sequence evolution, are given in a table shown as the next figure. (Source: Icko Iben, 1965, ApJ, 141)
Fig. 6.— Top panel: Table of times required for various stages of pre-main tracks evolution for stars with masses from 0.5 to 15 $M_\odot$. The tracks themselves are shown in the previous figure. Lower panel: variation with time of various interior parameters as a 1 $M_\odot$ star evolves down the Hayashi track. (Source: Icko Iben, 1965, ApJ, 141)
Fig. 7.— Isochrones for pre-main sequence evolution. The computations and the data are somewhat out of date, but give the general idea. (Source: Iben and Talbott, 1966)
Fig. 8.— HR diagram for young stellar objects in three regions of extensive recent star formation. The X-ray selected sample is shown, as well as the full set of known pre-main sequence stars in each of the three fields studied. (Source: Ramirez, Rebull, Stauffer et al, 2005, ApJ)
Fig. 9.— Left top panel: The radius as a function of time for a molecular cloud during the free-fall collapse. Left bottom panel: evolution of density as a function of time during free-fall collapse phase. Right panel: HR digarm for T Tauri stars (young pre-main sequence stars) (composite from various regions of star formation).
4. Main Sequence Details

The main sequence has already been discussed in class. The main feature is that the H-burning phase lasts a long time as a lot of energy is released during pp or CNO cycle burning of H into He. Furthermore, H is the most abundant element by far within a star. The phase ends when H is exhausted in the core of the star.

Details discussed in class include composition discontinuities, where $\mu$ is higher in the core due to previous or ongoing nuclear burning. Furthermore, at the boundaries of a convection zone $\mu$ can change abruptly. Since $T$ and $P$ must be continuous, when $\mu$ abruptly changes, $\rho$ must also abruptly change.

Also higher mass stars have convective cores which keep the H and He well mixed throughout the core, unlike lower mass stars with radiative cores. That means that higher mass stars have a larger fraction of their total H available for burning, and at the end of the MS phase, the He core includes more mass.

Also discussed effect of change in overall He or metal abundance in MS star. Effect of increasing He is that of increasing $\mu$, since He provides little opacity. $L \propto \mu^7$, so such a star has higher luminosity. The increase in $\mu$ just in the core of the star from H burning leads to a small increase in $L$ as the H burning progresses, thus stars become slightly more luminous than the “zero age main sequence”, i.e. the MS just at the time of H ignition. Increasing metals primarily affects the opacity, as their contribution to the total number density is too small to perceptibly alter $\mu$. 
Fig. 10.— Top panel: Distribution of H within a $1 \, M_\odot$ at different stages during core H-burning (main sequence). Bottom panel: The same during the main sequence phase for a $5 \, M_\odot$ star. (Source: Salaris & Cassisi, 2005)
Fig. 11.— Top panel: Tracks for low mass stars beginning at the zero age main sequence during the core H burning phase. Bottom panel: trend with mass coordinates of various quantities ($T$, $\rho$, $P$, $\epsilon$ from H burning, etc) within a $1 \, M_\odot$ main sequence star. (Source: Salaris & Cassisi, 2005)
5. Red Giants

This phase of the evolution of low mass stars is characterized by core contraction to reach $T$ high enough to initiate He burning, coupled with envelope expansion and H burning in a narrow shell. The envelope expansion causes cooling and the development of an extensive envelope convection zone, which constrains the star’s evolution roughly to the $L, R, T_{\text{eff}}$ region of the Hayashi track.

In this phase, once thermal equilibrium has been restored after the chaos that results when H is exhausted in the core, the dominant contribution to the luminosity is from shell H burning, and $L \propto \frac{dM_c}{dt}$.

The red giant phase is terminated by the ignition of He in the core of the star.

5.1. The Maximum Mass of an Isothermal Core

If a nuclear fuel is exhausted and $T$ is too low to start burning the next available nuclear fuel, then $\epsilon = 0$ inside the core, so $L$ must be 0 inside the core as well, which implies that $T$ must be constant inside the core out to $r_c$. Since $P(r)$ must rise rapidly to support the core against gravity, $\rho(r)$ must rise rapidly within the core. If the required $P(r)$ cannot be sustained, the core will be forced to contract by the weight of the overlying mass. The maximum mass for such an isothermal core with no nuclear energy generation within the core is given by the Schoenberg-Chandrasekhar limit. This is derived by matching the envelope pressure with the maximum possible pressure from the core at the interface between the core and the envelope, i.e. $r = r_c$. If we define $q$ as $M_c/M$, the fraction of the total mass in the core, then the condition as derived in class is
\[ q < \left( \frac{\mu_{\text{env}}}{\mu_c} \right)^2 \left( \frac{2}{3} \right)^2 \sqrt{\pi} = 0.28 \left( \frac{\mu_{\text{env}}}{\mu_c} \right)^2 \]

A better analysis gives \( q < 0.37 \left( \frac{\mu_{\text{env}}}{\mu_c} \right)^2 \).

In a red giant core, the material is mostly fully ionized He, so \( \mu_c = 4/3 \).

If \( M_c \) is too large, the core will contract and its central temperature will rise, until degeneracy steps in and the core can be stably supported against the pressure from the overlaying envelope.

If \( M_c > \) the Chandrasekhar limit, core contraction is not halted by the onset of degeneracy, and it continues until the next nuclear fuel is ignited.

### 5.2. Shell Burning

Burning of H occurs in a thin shell around the inert He core.

"Facts": 1) An active shell burning source tends to remain fixed in radius with time. The mass of the core increases as burning proceeds outward in mass, but \( r_c \) remains fixed.

2) As an active shell burns outward, as the core contracts, the envelope expands and the stellar radius \( R \) increases.

The second "fact" is supported by the rough argument given in class about conserving total gravitational energy \( \Omega \), while moving a small mass inwards from \( r_c \) and the same amount of mass outwards from \( r_c \). A much larger outward motion is required to keep the total \( \Omega \) constant.
5.3. Models of H-Burning Shell Sources

Simple models of a star with a shell source good enough to characterize their behavior, although not adequate in detail, can be constructed if one is willing to make a number of assumptions. There are: a) adopting a model for the envelope that \( \rho(r) = \rho_s (r_s/r)^3 \) and \( T(r) = T_s (r_s/r) \). The second was proved (approximately) in class. b) adopting a linear density law for the core, \( \rho(r) = \rho_c (1 - r/r_s) + \rho_s \approx \rho_c (1 - r/r_s) \). The energy generation rate is parameterized as \( \epsilon = \epsilon_0 \rho T^n X^2 \) for \( r > r_s \) and \( \epsilon = 0 \) inside \( r_s \). The \( X^2 \) appears in \( \epsilon \) as it is H-burning. If it is via the CNO cycle, only \( X \) appears, for \( p-p \) burning, \( X^2 \) is required.

From this we can infer the following relations for the core from the basic equations of stellar structure, integrating in the case of the first one given below between \( r_s \) and \( R \):

\[
r_s = \left[ \frac{3M_c}{\pi \rho_c} \right]^{1/3} \quad \text{and} \quad \frac{r_s}{R_\odot} = 0.18 \left[ \frac{M_c/M_\odot}{\rho_c/10^3} \right]^{1/3}
\]

\[P_c = P_s + \frac{56}{36} G \rho_c^2 r_s^2 \]

If the core is degenerate, it is assumed to be isothermal.

\[
T_c = \frac{5\pi G}{36} \frac{\rho_c r_s^2 \mu_m H}{k} = \left[ \frac{M_c}{M_\odot} \right]^{2/3} \left[ \frac{\rho_c}{10^3} \right]^{1/3}
\]

For the envelope, assuming that most of the mass is inside \( r_s \),

\[
R = \frac{M_s - M_c}{4\pi \rho_s r_s^2}
\]

\[T_s \approx 8 \times 10^6 \mu_{env} \frac{M_c}{M_\odot} \frac{R_\odot}{r_s} \approx 4 \times 10^8 \mu_{env} \left[ \frac{M_c}{M_\odot} \right]^{2/3} \left[ \frac{\rho_c}{10^3} \right]^{1/3} K.
\]
\[
\frac{R}{R_\odot} = 1.5 \times 10^{-2} \left( \frac{M}{M_\odot} \right)^{(10\eta-21)/27} (1 - q)^{5/3} q^{(10\eta-66)/27} \left( \frac{\rho_c}{10^3} \right)^{(5\eta+6)/27}
\]

\[
\frac{L}{L_\odot} = 39.5 \left( \frac{M_c}{M_\odot} \right)^{(21+2\eta)/9} \left( \frac{\rho_c}{10^3} \right)^{(3+\eta)/9}
\]

Note that \( T_{\text{eff}} = L/(4\pi R^2) \) and so can be derived from the last two equations.

\[
\frac{d\log(L)}{d\log(T_{\text{eff}})} = -\frac{12(\eta + 3)}{7\eta + 3} = -1.98 \quad \text{for CNO cycle, } \eta = 16
\]

so as \( \rho_c \) increases as the burning proceeds, the luminosity increases and \( T_{\text{eff}} \) decreases.

It is also possible to evaluate the timescale for core contraction (\( \alpha \) in the formula below is the form factor for the gravitational potential energy):

\[
\Delta t = \Delta E(\text{core})/L_c \approx 5.1 \times 10^{10} \alpha X_{\text{env}} \frac{M_c}{M_\odot} \frac{L_\odot}{L_s} \left[ 2 \frac{\Delta M_c}{M_c} - \frac{\Delta R_c}{R_c} \right] \text{ years},
\]

so that a 10% change in \( M_c \) or \( R_c \) for a 5\( M_\odot \) star would take about 10^6 yr.

To evolve models in this phase, increase \( \rho_c \) with time as the H burning proceeds and increase \( M_c \) slightly with time for the same reason.

### 5.4. Useful Characteristics of RGB Stars

The tip of the RGB for old low mass stars lies at an approximately constant luminosity, with only a slight dependence on metallicity. It can thus be used as a distance indicator for nearby galaxies, where individual stars can be resolved and a HR diagram constructed to find the apparent luminosity of the RGB tip.
The RGB bump is an increase in the number density of stars along the RGB as a function of $L$ within a narrow range of luminosity. It occurs due to a slow down/crossover in the speed of stellar evolution along the RGB, i.e. in $dL/dT$, when the H burning shell reaches the discontinuity in $\mu$ caused by the maximum extent of the envelope convection zone, which at times reaches down to material where the H has been burned.

For old low mass stars, the RGB tracks become cooler (redder) as the metallicity increases, providing a useful diagnostic of metallicity for simple old stellar populations such as globular clusters or many elliptical galaxies.
Fig. 12.— Tracks in the H-R diagram for stars of various masses from the zero age main sequence to core He ignition. (Source: Icko Iben, Ann Revs.)
Fig. 13.— Isochrones and tracks for stars of various masses from the zero age main sequence to core He ignition. (Source: Icko Iben, Ann Revs.)
Fig. 14.— Left panel: Tracks for stars of various masses from the zero age main sequence to core He ignition. Right panel: details in the evolution of a 10 $M_\odot$ star. (Source: Icko Iben, Ann Revs.)
Fig. 15.— Predicted isochrones for an old (age 13 Gyr) simple stellar population in the regime of the RGB, shown for various metallicities. The discrete points represent the loci of observations of selected Galactic globular clusters. Note the good agreement of the observed and predicted location of the RGB tip. (Source: Fig 13 of VandenBerg, bergbusch & Dowler, 2006, ApJS)
Fig. 16.— Integrated count down to an observed $K$ mag (top panel) and differential luminosity function (i.e. counts in mag bins) (lower panel) for RGB stars in the Galactic globular cluster NGC 6441. The RGB bump is obvious in the lower panel. (Source: Valenti, Ferraro & Origlia, 2004, MNRAS, 354, 815)
Fig. 17.— JHK absolute magnitudes at the RGB tip as a function of metallicity for a set of Galactic globular clusters. (Source: Valenti, Ferraro & Origlia, 2004, MNRAS, 354, 815)
Fig. 18.— Bolometric magnitudes at the RGB bump (left panel) and RGB tip (right panel) as a function of metallicity for a set of Galactic globular clusters. (Source: Valenti, Ferraro & Origlia, 2004, MNRAS, 354, 815)
6. Helium Flash

Ignition of a new fuel in a classical gas causes a rapid rise in $T$, also in $P$, hence the core expands outward, this cools the core and stabilizes the nuclear burning.

Ignition of a new fuel in a degenerate gas gives rise to a thermal runaway. As the fuel begins burning, $T$ rises, but $P$ is not sensitive to $T$, but rather to $\rho$. Hence the burning accelerates, as $\epsilon \propto \rho T^n$ and $n$ for He burning is high, $n \sim 40$ for $T \sim 10^8$ K, much higher than that of the CNO cycle of H burning. This spiral of higher $T$ leading to much higher $\epsilon$ and thus still higher $T$ continues until the degeneracy is lifted.

The huge core luminosity ($L \sim 10^{11} \, L_\odot$) during the short timescale (a few seconds) of the He flash does not reach the surface.

A star of mass $M < 0.6 \, M_\odot$ cannot become hot enough for He ignition before the H-burning shell burns through almost the entire star. He never ignites in such a star. A star of mass 0.6 to $2M_\odot$ will ignite He in a degenerate core, and thus undergo the He flash. The core of a star of mass 2 to $6M_\odot$ will exceed the Schoenberg-Chandrasekhar limit before it becomes degenerate. Such a core then loses isothermality, contracts, and $T_c$ increases rapidly. He ignites at the tip of the RGB. A star of mass $M \gtrsim 6 \, M_\odot$ has a convective core when on the main sequence, so the region of H exhaustion at the end of the MS phase is very large, the core is large, and $q > q(CS)$. So the core is never isothermal, it must contract as soon as H is exhausted there. $T_c$ rapidly rises, reaches $\sim 10^8$ K, and $3\alpha$ He burning begins. This core is never degenerate, so there is no He flash.

The low density extended envelope is not tightly bound and mass loss rates increase along the RGB. Low mass giants may in some cases lose so much mass that their core can never become hot enough to ignite He. Once all of the envelope mass has been lost, the star leaves the RGB as the degenerate He core continues contracting to become a He white dwarf.
7. Helium Core Burning

The star, after a somewhat chaotic period immediately after He ignition, reaches the zero age horizontal branch. This phase is characterized by He burning in the core and H burning in a shell. The core is convective due to the high $T$ dependence of He burning. The key parameters are thus the core mass and the envelope mass. The larger the envelope mass, the hotter the H burning shell. The ratio of energy production via He burning to that from H burning is a key parameter.

The luminosity of the star is fixed primarily by the mass of its He core. This is approximately constant for stars with age $> 4$ Gyr, i.e. initial mass lower than $1.5 M_\odot$. Thus in the HR diagram, the stars form an approximately horizontal locus, from which the name “horizontal branch” comes. This approximately constant luminosity means that rough distance estimates can be obtained based on stars in parts of the horizontal branch.

The core He burning locus crosses the instability strip, producing radially pulsating RR Lyrae variables for low mass stars, and Cepheid variables among higher mass He core burning stars. We can make use of the period – luminosity relation (see the class note on helioseismology and asteroseismology) for each of these classes of pulsating stars so these variable stars can be viewed as standard candles for extragalactic distance determinations. However, the RR Lyrae variables can only be detected even with HST in the very nearest galaxies, as they are fainter than Cepheid variables.

$T_{\text{eff}}$ for a HB star of fixed $M$ and $M_c$ depends on the envelope mass. Stars with higher envelope mass have lower $T_{\text{eff}}$. The population along the HB, i.e. the range of envelope mass, depends on the initial star mass and its mass loss history, and the latter may vary from star to star depending on many details we have ignored such as rotation rates.

The core He burning phase has short lifetime as there are not as many He atoms to be
burned due to higher mass than H per atom, and the amount of energy per He burned is much lower than that from combining 4 protons into 1 He nucleus (\(\sim 10\%\) as large per unit mass). Furthermore the stellar luminosity on the HB for a star of given mass is higher than when the star is in the H burning main sequence phase.

Evolution off the ZAHB depends on the details of the relative contribution of the H burning shell and the He core to the total energy generation.

When core He is exhausted, the core consists large of C and O.
Fig. 19.— The HB in old Galactic globular clusters. The MS and RGB sequences are fiducial points, while for the HB region, each star is shown. (Source: Vandenbergh, 2000, ApJS, 129, 315)
Fig. 20.— The upper RGB, AGB and HB in the old Galactic globular cluster M5. (Source: Sandquist & Bolte, 2004, ApJ, 611, 323)
8. Asymptotic Giant Branch

AGB is characterized by the presence of two burning shells. A C-O core, then a He burning shell, then a layer of inert He, then a H burning shell, then an envelope.

The luminosity is determined only by the core mass, not by the total mass of the star. The luminosity is from the H burning shell, which is affected primarily by the core mass. The pressure drops very steeply just outside the H burning shell, and thus low mass envelope does not affect the properties of the burning shell.

A core mass - luminosity relation was first derived by Paczynski. It can be obtained using a homology (dimensional) analysis with $M_C$ and $R_C$ (the mass and radius of the core) as variables. Electron scattering is used as the opacity. A limit for low $M_c$ is derived assuming radiation pressure is not important. The high $M_c$ limit is derived assuming the radiation pressure dominates.

The result given in class for large core masses is the Paczynski relation:

$$\frac{L_C}{L_\odot} = 5.92 \times 10^4 \left[ \frac{M_C}{M_\odot} - 0.52 \right]$$

while for small core masses it was $L_C \propto M^7 R_C^{-16/3}$.

(This calculation is outlined in Sections 32.2 and 33.5 of KW.)

There are no nuclear reactions in the C-O core, which thus contracts and heats up. As this happens, the envelope expands, cools, becomes convective, and goes back to the RGB track (the Hayashi region) as a second ascent red giant (AGB star). Since the $T$ for ignition in a C-O core is very high, the star must contract even more to get to such a high temperature, so it climbs even higher in $L$ than does a first ascent red giant (and slightly cooler, but not much, recall how steep the Hayashi line is).
He shell burning is very unstable due to the extreme sensitivity of the energy generation rate for this process to $T$, $\epsilon \propto \rho T^{40}$. The shell is thus very thin. Nuclear reactions in in such a thin He-burning shell easily lead to thermal instabilities.

The H burning shell is on most of the time, while the He burning shell is off most of the time. The layer of inert He grow with time as the H burning shell processes material, with no energy supply it contracts, heats up, and He at is base eventually ignites, but in degenerate material. This produces a mini-He flash from the ignition which expands out the H burning shell, cools it, and turns it off. This is called a thermal pulse. The cycle repeats.

The instability of He shell burning in this double shell case gives rise to a long series of thermal pulses, which in turn give rise to convective zones that are associated with each thermal pulse, then subsequently retreat. The double shells, their on/off cycles, and flashes that induce convection leads to mixing (dredge up) of material from the He burned region into the envelope, and introduction of protons into hot regions where all initial protons have already been burned. Neutrons are produced in these peculiar circumstances through reactions which do not occur in earlier stages of stellar evolution. This gives rise to the $s$-process of neutron capture, which produces many of the isotopes past Fe using the Fe-peak nuclei as seeds.

The dredge up is substantial, and can change the envelope composition from the normal Solar ratio $n(C)/n(O) \sim 1/3$ to $n(C)/n(O) > 1$, a change in abundance which produces strong effects on the emitted spectrum through the different molecules that dominate in the stellar atmosphere and hence the molecular bands that appear in these cool supergiants in the two cases.

Stars leave the AGB when their H-rich envelopes are totally gone, most commonly lost through mass loss by stellar winds, more rarely by the H-burning shell burning almost
entirely through the star.

Only stars more massive than $\sim 1.4 \, M_{\odot}$ can achieve the high temperatures necessary to ignite a C-O core.
Fig. 21.— Thermal pulses in the He shell source of a 5 \( M_\odot \) AGB star. (Source: Kippenhahn & Weigart, pg 333)
Fig. 22.— Details of flashes in a $7 M_\odot$ AGB star. (Source: Iben & Renzini, ARAA, 1983, 21, 271)
Fig. 23.— More modern calculations of thermal flashes in a27 $M_\odot$ AGB star. (Source: Salaris & Cassisi, 2005)
Fig. 24.— Details of locations of convection zones before, during and after He shell flashes in a 7 $M_\odot$ AGB star. This leads to dredging up from the interior to the surface of material processed through H burning.
Fig. 25.— AGB tracks showing the instability strip for pulsation in the first overtone (these are long period Mira variables), the Eddington limit, and the planetary formation line (pulsation in the fundamental mode, which results in subsequent rapid ejection of the entire envelope). Note the very high luminosities. (Source: Iben & Truran, 1978, ApJ, 220)
In following the progression of our models we also compute the period for pulsation in the first harmonic radial mode using

\[ P(\text{days}) \approx 0.038 R_*/M_* \] (32)

as given by Wood (1976) and determine the characteristics of the star at the onset of Mira variability and at the occurrence of planetary nebula ejection, following the prescriptions of Wood and Cahn (1977).

IV. EVOLUTION IN THE M-L PLANE AND IN THE H-R DIAGRAM

Evolutionary tracks in the mass-luminosity (M-L) plane are shown in Figure 1. The number beside each track is the initial mass in solar units. It has been assumed that 0.2 \( M_\odot \) is lost by each model star during its first ascent of the giant branch (Iben and Renzini 1975a, b). Dashed curves in Figure 1 are loci of constant period for pulsation in the first harmonic mode. We have assumed that when a track reaches the straight line of larger positive slope, Mira variability is initiated, and that when a track reaches the straight line of smaller positive slope, a planetary nebula is ejected. The ejected nebula is assumed to contain all of the matter between the location of the hydrogen-burning shell and the surface at the time of ejection.

We emphasize that the location of the Mira instability line and the location of the planetary nebula

![Diagram of evolutionary tracks in the H-R diagram](image)

Fig. 26.—The tracks of AGB stars, with PN formation and Carbon star formation indicated. (Source: Iben & Truran, 1978, ApJ, 220)
9. Planetary Nebulae

Mass loss is very important in AGB stars due to their high luminosity and very extended envelopes. Recall that the escape velocity $v_{\text{esc}}$ from the surface of a star is $\sqrt{2GM/R}$. If a fraction $f$ of the total emitted radiative momentum passing through the atmosphere is absorbed in there and is used to drive a stellar wind, then the mass loss rate in a time interval $\delta t$ is given by:

$$\frac{dM}{dt} \delta t \, v_{\text{esc}} = f \frac{L}{c} \delta t$$

so

$$\frac{dM}{dt} = f \frac{v_{\text{esc}}}{c} \frac{LR}{GM}$$

Note the dependence on $LR/M$; this factor is very large in AGB stars.

The material actually accelerated is mostly the dust grains in the very cool outer atmosphere of the AGB star. Typical mass loss rates for giants are $\sim 10^{-9}$ to $10^{-6} \, M_\odot/\text{year}$. The empirical formula for mass loss developed by Reimers is often used to parameterize the above relationship for radiatively driven winds. It takes the form:

$$\frac{dM}{dt} = 10^{-13} \frac{L}{L_\odot} \frac{R}{R_\odot} \frac{M_\odot}{M} \frac{M_\odot}{\text{yr}}.$$ 

Once the star begins pulsating in the fundamental mode (Mira variables pulsate largely in the first overtone), a superwind with much higher mass loss rates, $\sim 10^{-4} \, M_\odot/\text{year}$, develops. The remaining envelope of the star is rapidly lost. This marks the end of the AGB phase.

The star is now basically an exposed inert stellar core. No nuclear reactions are occurring in the core. The star is very hot ($T_{\text{eff}} \sim 100,000 \, \text{K}$) and very luminous. There is initially some shell burning around the core which is later extinguished. At this point the core of the star then becomes a very hot cooling white dwarf with no substantial nuclear
energy production and it rapidly cools.

Mass loss occurs so rapidly in the superwind phase that the planetary nebula nucleus (i.e. the star) becomes surrounded by clouds of gas ejected during the superwind phase, i.e. a planetary nebula. The star is, at least initially, still hot enough and emitting a high enough luminosity, to ionize the nebula. As the star starts cooling, eventually the nebula expands and disperses. The timescale for the fading away of the nebula is about $10^4$ yr.
Fig. 27.— The predicted distribution of planetary nebula nuclei in the log \( L \)–log \( T_{\text{eff}} \) plane. Tracks of stars of various masses from 0.55 to 1.4 \( M_\odot \) are shown, with the shading indicated the expected distribution. (Source: Kaler, Ann. Revs., 23, 1985)
10. White Dwarfs

The overall structure of white dwarfs, the importance of degeneracy, the mass – radius relation, and the Chandrasekhar limit, were covered earlier in this class. We do not revisit these points, although there are some figures illustrating the mass – radius relation as obtained from white dwarf binary systems and contrasting this with non-degenerate cool low mass stars (M dwarfs).

Since white dwarfs are the dying embers of low mass stars, there should be equality between the death rate of stars of the appropriate mass range and the birth rate of the associated compact object. This could be investigated by looking at the number of low mass stars formed in the galaxy as a function of time, the luminosity function of low mass field stars in the region of the Sun, and the spatial number density of the compact object appropriate for that range of initial stellar mass.

Such a calculation works out OK if one assumes for white dwarfs they are the remnants of all stars of 1 to 4 $M_\odot$ ever formed in the galaxy, and for neutron stars the assumed relevant mass range is 4 to 10 $M_\odot$. Stars less massive than $\sim 0.8 \ M_\odot$ (the exact value depends somewhat on the metallicity) have such long main sequence lifetimes that they have not yet become white dwarfs, even if formed 12 Gyr ago in the young proto-galaxy.

10.1. White Dwarf Cooling

We assume the white dwarf star is a polytrope of index 1.5, i.e. $P \propto \rho^{5/3}$, non-relativistic degeneracy. Because the interior is degenerate and cools by conduction, one can work out the heat loss from the star. Conduction by degenerate electrons is very efficient and hence the temperature gradient within the core is very low. Let $T_b$ be the temperature at the boundary between the degenerate core and the thin envelope. One can show that
\( (T_c - T_b)/T_c < 1/8 \) for a 0.5 \( M_\odot \) white dwarf with \( L = L_\odot / 400 \). So the degenerate core is almost isothermal and it contains almost all of the mass of the star. We assume \( T_b \approx T_c \) for the rest of the discussion of white dwarfs.

We assume radiative transfer of energy in the thin envelope, with Kramer’s opacity, \( \kappa \propto \rho T^{-3.5} \). We then evaluate \( T_b \) by integrating the equation of radiative equilibrium inward from the surface, and again by the degeneracy condition since this is the edge of the degenerate core.

The total thermal energy in the ions, whose mean atomic weight is \( A \), is

\[
U = \frac{3}{2} k T \frac{M}{\mu m_H} \approx 2 \times 10^{47} T_7^{12} \frac{12}{A} \text{ ergs.}
\]

We next wish to calculate the luminosity. We assume radiative transfer of energy in the thin envelope, with Kramer’s opacity, \( \kappa \propto \rho T^{-3.5} \). The radiative flux is then

\[
F = \frac{4}{3} \frac{a c T^3}{\kappa \rho} \frac{dT}{dr}
\]

Hydrostatic equilibrium applied in the thin atmosphere is then

\[
\frac{dP}{dr} = -\rho g, \quad g = G \frac{M}{R^2}
\]

\[
F = \frac{4}{3} \frac{a c T^3 g}{\kappa} \frac{dT}{dP}
\]

\[
F \propto \frac{g T^{7.5}}{P \rho}
\]
but using the non-relativistic degeneracy relations for a gas we find \( P \rho \propto T^4 \). Also for white dwarfs, if the mass is specified, the radius is known from the non-relativistic degenerate polytrope solution, \( R \propto M_c^{-1/3} \). So

\[
F \propto g T^{3.5} \propto \frac{M T^{3.5}}{R^2}, \quad L = 4\pi R^2 F \propto M T^{3.5}
\]

\[
L \propto T_0^{3.5} M \quad L \approx 2 \times 10^{-3} T_7^{3.5} \left( \frac{M}{M_\odot} \right) L_\odot
\]

We look for a cooling timescale, which we obtain since there is no energy source in a white dwarf, no gravitational contraction, just radiation lost from cooling interior. Let \( t = t_0 \) when the star reaches the white dwarf stage. Then:

\[
L = - \frac{dU}{dt} = - \frac{3}{2} \frac{k}{\mu(\text{ions}) m_H} M \frac{dT_c}{dt} = - \frac{3}{7} \frac{k}{\mu(\text{ions}) m_H} M \frac{T_c}{L} \frac{dL}{dt}.
\]

\[
- \frac{dL}{dt} \propto M T_c^6
\]

Thus the cooling rate is much more rapid for the hotter white dwarfs. The cooler ones cool more slowly.

\[
T_c^{3.5} \propto - \frac{dT_c}{dt}, \quad (t - t_0) \propto T_c^{-5/2}, \quad (t - t_0) \approx 4.5 \times 10^7 \left[ \frac{M/M_\odot}{L/L_\odot} \right]^{5/7} \text{ yr}
\]

We can thus predict the luminosity function for a population of cooling white dwarfs as a function of \( L \) or \( T_c \) given the initial mass function and the star formation history of the population. The calculation is straightforward for a population formed in a single burst (all stars have the same age).
This simple calculation ignores crystalization and other exotic phenomena that may occur at very high density which would affect the specific heat of the degenerate core.

10.2. White Dwarf Taxonomy

White dwarfs are divided into various classes depending on the appearance of their spectra, i.e. on the composition of the envelope. This depends on how far the star progressed in terms of nuclear burning (He burning, or just H burning) before the star became a white dwarf and on how much mass was lost from its envelope during the AGB phase.

DA white dwarfs show lines from H, DB white dwarfs show no H, but He is strong.

It was recently discovered that a small fraction of white dwarfs show absorption lines of metals in their envelopes. These are called DZ white dwarfs, with DAZ and DBZ being the full classification, depending on the presence or absence of hydrogen lines. This is completely unexpected as the surface gravity of white dwarfs is very high, they are very compact. Gravitational settling rapidly drags all the heavy elements into the interior of a white dwarf. Speculation that this is the result of accretion of comets, proto-planets, etc exists in the literature. Accretion of the normal interstellar material does not appear to be enough to account for the fraction of WDs that show metal lines and the amount of metals they are inferred to have in their atmospheres.
Fig. 28.— The contributions of various terms to the total luminosity of a cooling 0.55 $M_\odot$ white dwarf. (Source: D’Antona & Mazzitelli Ann. Revs., 28, 1990)
Fig. 29.— Radius as a function of mass for white dwarfs. (Source: Schmidt, 1996, A&A, 311, 852)
Fig. 30.— Initial – final (WD) mass relation for low and intermediate mass stars, and (bottom) WD evolutionary tracks as compared to main sequence. (Source: Salaris & Cassisi, 2005)
Fig. 31.—Lower left panel: Color-magnitude diagram for the old and nearby globular cluster M4 showing the lower main sequence and the white dwarf sequence. Lower right panel: field stars (mostly not members of M4) Photometry is from WFPC2/HST images. Top panels: the selection of cluster members through proper motions (units are WFPC2/HST pixels over the 6 year baseline between the two sets of images). The proper motion of the globular cluster M4 is obvious as it is moving coherently in a manner quite different from that of the average field star in this direction. (Source: Fig. 3 of Richer, Fahlman, Brewer et al, 2004, AJ)
Fig. 32.— Color-magnitude diagram for the old and very nearby globular cluster NGC 6397 showing the main sequence turnoff region, the main sequence, and the white dwarf sequence. Photometry is from ACS/HST images. (Source: Hansen, Anderson, Brewer et al, 2007, ApJ in press)
Fig. 33.— Absolute visual magnitudes as function of log($T_{\text{eff}}$) for DA white dwarfs from the PG catalog. Dashed and solid lines denote two sets of theoretical isochrones, with corresponding masses indicated in the figure. Isochrones are labelled in units of $\text{lg}(\tau)$, where $\tau$ is the white dwarf cooling time in years. (Source: Fig. 15 of Liebert, Bergeron & Holberg, 2005, ApJS)
Fig. 34.— Mass distribution for a large sample of field white dwarfs. (Source: Fig. 2 of Bergeron, Gianninas & Boudrealt, 2007, ASP Conference, proceedings of 15th European White Dwarf Workshop)
Fig. 35.— Mass – radius relation for white dwarfs and for M dwarfs. (Source: Fig 11 of Maxted, Marsh et al, 2005, MNRAS, paper on an eclipsing white dwarf - M dwarf binary)
Fig. 36.— Luminosity function for white dwarfs, sample from the SDSS. (Source: fig. 7 of DeGennaro, von Hippel, et al, 2007, AJ)

High mass stars can reach the $T$ necessary for He ignition in their central regions before the core becomes degenerate, so there is then no He flash. At this time such stars rapidly become supergiants, and their subsequent evolution is manifested in the HR diagram largely by loops at approximately fixed $L$ to higher and lower $T_{eff}$ (i.e. blue and red supergiants). The electrons in the core do not become degenerate until the final burning stage, when the core consists of iron.

Because of their high luminosity, mass loss plays an important role in the evolution of massive stars. If almost the entire envelope is lost, the star is basically just a very hot core with a very thin H-poor envelope. Such a star is called a Wolf-Rayet star. It may have show in its atmosphere highly enhanced He or N, from CNO process burning of H, or the thin atmosphere can even consist most of C and O, when even the inert He shell is removed through mass loss.

Neutrino losses become important for the later stages of nuclear burning that occur in high mass stars.

The lifetimes of high mass stars in all phases of evolution, especially beyond He burning, are very short; the nuclear fuels are very rapidly consumed.

$L$ for high mass stars, even on the main sequence, is perilously close to the Eddington limit, $L_{(Edd)} \sim 3.2 \times 10^4 \ (M/M_\odot) \ L_\odot$. Throughout their evolution, $L$ cannot increase much without the star losing substantial mass and becoming unstable.

Rather than dying as cooling white dwarfs, massive stars explode as supernovae, often returning a large fraction of their mass to the ISM. The ejected mass contains material has been processed through various nuclear burning stages and is highly enriched in heavy elements.
Fig. 37.— Tracks in the H-R diagram for stars of various masses from the zero age main sequence to core He ignition. (Source: Icko Iben, Ann Revs.)
The evolution of a 25 \( M_\odot \) star from the main sequence up to the onset of the iron core collapse

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Abstract

We present the evolution of a 25 \( M_\odot \) star model having solar chemical composition from the pre-main-sequence phase up to the onset of the iron core collapse. This evolution has been obtained with the latest version (release 4.0) of our hydrostatic evolutionary code FRANEC, and hence it is totally homogeneous with all of our stellar evolutions concerning low- and intermediate-mass stars. A network including 149 nuclear species has been fully coupled to the equations describing the temporal variations of the physical quantities. No quasi-nuclear statistical equilibrium approximation has been adopted up to \( 4 \times 10^8 \) K. All of the evolutionary properties of the stellar model are discussed in detail. Whenever possible, comparisons with similar evolutions available in the literature have been performed. An iron core mass smaller than the values recently quoted in the literature for the same mass and solar chemical composition has been obtained (\( M_{\text{Fe}} = 1.513 \ M_\odot \)).

Fig. 38.— The evolution of a 25 \( M_\odot \) star from the main sequence to the onset of final core collapse. (Source: Chieffi, Limongi & Straniero, 1998, ApJ, 502, 737)
Fig. 39.—Selected details of the evolution of a 25 $M_\odot$ star during advanced burning phases.
12. Issues that have been Ignored

rotation - see Maeder & Meynet, 2000, Annual Rev Astron. Astrophys., 38, 143 for a review, and their more recent papers if interested